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LETTER TO THE EDITOR

On the validity of Dirac's conjecture regarding first-class secondary constraints

Mark J Gotay

Department of Mathematics and Statistics, The University of Calgary, Calgary, Alberta, Canada T2N 1N4

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Abstract. Two examples are presented which clarify Dirac's conjecture that 'all first-class secondary constraints generate gauge transformations.' In the general case the status of this conjecture is ambiguous, depending crucially upon the physical interpretation of the Lagrangian at hand. However, I propose a consistent general method of interpreting Lagrangians such that, relative to this interpretative framework, Dirac's conjecture holds *true*.

In his famous work on degenerate Lagrangian systems, Dirac (1964) conjectured that all first-class secondary constraints generate 'gauge' transformations which leave the physical state invariant. Recently, Cawley (1979) claims to have found systems for which this conjecture fails. In this article I present two examples which cast doubt upon this claim and indicate that a closer examination of Dirac's conjecture is warranted. A proper understanding of this conjecture requires a detailed study of the symplectic geometry underlying classical mechanics (Gotay *et al* 1978, Gotay and Nester 1979a, Gotay 1979); this work (with J Nester) will be published elsewhere. Throughout this paper I utilise only the standard canonical analysis (see Dirac 1964).

Consider a degenerate Lagrangian system. Upon going over to the Hamiltonian formalism, one finds that the dynamics of the system is determined by the 'total' Hamiltonian

$$H_T = H + u_A\phi^A, \tag{1}$$

where H is the usual Hamiltonian, u_A are Lagrange multipliers, and $\phi^A(q, p) \approx 0$ are primary constraints ($\approx$ denotes 'weak' equality). To ensure a physically well defined evolution, one implements Dirac's well known 'constraint algorithm' by demanding that the constraints be preserved in time. These consistency conditions determine certain of the u_A and also give rise to further (i.e., secondary) constraints which must be preserved as well.

A constraint is *first class* provided its Poisson bracket with every other constraint weakly vanishes, and *second class* otherwise. Dirac showed that the constraint algorithm determines the multipliers of the second-class primary constraints in (1) while leaving arbitrary the coefficients of the first-class primary (FCP) constraints. Consequently, the latter are generating functions of motions which leave the physical state invariant (i.e., *gauge transformations*). Let G_1 denote the set of all FCP constraints, and consider the hierarchy

$$G_{i+1} = G_i + \{G_i, G_i\} + \{G_i, H_T\}.$$

This process must terminate with a set $G \subseteq FC$, where FC denotes the set of all functionally independent first-class constraints. Since elements of G_1 generate gauge transformations, so must elements of G.

Apparently, Dirac knew of no example in which the equality

$$G = FC \qquad (\textit{Dirac's test})$$

—assuring that *all* first-class constraints generate gauge transformations—did not hold, although he could not prove it in the general case. This led Dirac to conjecture that first-class *secondary* (FCS) constraints also generate physically irrelevant motions and hence should be included in the Hamiltonian as well. Dirac therefore proposed adjoining the FCS constraints ψ^a with arbitrary multipliers ω_a to H_T thereby obtaining the 'extended' Hamiltonian

$$H_E = H_T + \omega_a \psi^a. \tag{2}$$

Thus, Dirac reasoned that H_E would give the most general evolution of the system.

Numerous examples are now known for which Dirac's test fails (Cawley 1979, Gotay *et al* 1978, Gotay and Nester 1979a, Frenkel 1980, Allcock 1975, 1980). In fact (see Gotay and Nester 1979a and Gotay 1979) it can be shown that $G = FC$ iff $d\psi \not\approx 0$ for every FCS constraint ψ produced by the constraint algorithm. However, the significance of the failure of Dirac's test has not been made clear in the general case. Cawley (1979) asserts that this failure signals the presence of FCS constraints which do not generate gauge transformations, in the sense that H_E and H_T give rise to physically inequivalent evolutions. In other words, Cawley equates the failure of Dirac's *test* with the failure of Dirac's *conjecture*.

Lorentz-gauged electromagnetism is a system for which Dirac's test fails and for which Cawley's assertion is incorrect. The Lagrangian is

$$L = \int [-\tfrac{1}{4}F_{\mu\nu}F^{\mu\nu} + \tfrac{1}{2}\lambda^{-1}(\partial_\mu A^\mu)^2]\, d^3x \tag{3}$$

where λ is a Lagrange multiplier. Transforming to the Hamiltonian description, one obtains

$$\pi^0 = \lambda^{-1}(\dot{A}_0 - \partial_k A^k), \qquad \pi^k = \dot{A}^k - \partial^k A_0$$

along with a FCP constraint $\pi^\lambda \approx 0$; the total Hamiltonian is

$$H_T = \int [\tfrac{1}{2}\pi^k\pi_k + \pi^k\, \partial_k A_0 + \tfrac{1}{2}\lambda\, (\pi^0)^2 + \pi^0\, \partial_k A^k + \tfrac{1}{4}F_{lm}F^{lm} + u\pi^\lambda]\, d^3x.$$

The constraint algorithm produces two FCS constraints $-\tfrac{1}{2}(\pi^0)^2 \approx 0$ (equivalent to $\pi^0 \approx 0$) and $\partial_k \pi^k \approx 0$; the field equations are

$$\dot{\lambda} = u, \qquad \dot{\pi}^\lambda = \dot{\pi}^0 = 0, \qquad \dot{\pi}^k = \partial_l F^{kl}$$
$$\dot{A}_0 = \partial_k A^k, \qquad \dot{A}_k = \pi_k + \partial_k A_0.$$

As the FCS constraint $\psi = -\tfrac{1}{2}(\pi^0)^2$ satisfies $d\psi \approx 0$, Dirac's test fails. Since ψ is ineffective† one may, following Cawley (1979), drop the term $\lambda\psi$ from H_T. Then one

† In the sense that it generates no motion whatsoever.

calculates $G=G_1=\{\pi^\lambda\}$, whereas $FC=\{\pi^\lambda, \pi^0, \partial_k\pi^k\}$. However, the extended Hamiltonian

$$H_E = H_T + \int [\omega_1\pi^0 + \omega_2(\partial_k\pi^k)]\,d^3x$$

gives rise to field equations identical to the above, except for the last two which are replaced by

$$\dot{A}_0 = \partial_k A^k + \omega_1, \qquad \dot{A}_k = \pi_k + \partial_k A_0 - \partial_k\omega_2$$

respectively. These equations are clearly consistent with the known gauge freedom of the electromagnetic field. Despite the failure of Dirac's test, the FCS constraints π^0 and $\partial_k\pi^k$ unquestionably generate gauge transformations (which, however, do not respect the Lorentz gauge condition). Thus, as Dirac surmised, replacing H_T by H_E simply reveals the full gauge-transforming power of the electromagnetic field.

This example suggests that the failure of Dirac's test should properly be regarded as a consequence of 'built-in' gauge conditions. Indeed, consider a Lagrangian in which one has deliberately fixed a gauge. The motions generated by the FCP constraints preserve this built-in gauge condition and, since G is generated by G_1 and H_T, the motions generated by elements of G must also preserve the gauge condition. Furthermore, note that Dirac's test does not fail for theories where gauge conditions have not been imposed *a priori* (in particular, ordinary electromagnetism). From this standpoint, then, every first-class constraint generates a gauge transformation, but only those FCS constraints which are contained in G generate motions which preserve the built-in gauge conditions.

Thus, the failure of Dirac's test need *not* imply the failure of Dirac's conjecture. On the other hand, Cawley's assertion *is* valid provided one adopts an 'unusual' physical interpretation of the Lagrangian at hand. For example, one could conceivably *interpret* (3) as the Lagrangian, not for Lorentz-gauged electromagnetism, but rather for a massless divergence-free spin-1 field. In this case H_T and H_E are certainly not equivalent, and this is consistent with the claim that π^0 and $\partial_k\pi^k$ do not generate gauge transformations. Since there does not appear to be any obvious reason why this alternative interpretation of (3) should be discounted, Dirac's conjecture fails for the massless divergence-free spin-1 field.

The correctness of Dirac's conjecture therefore depends in an essential way upon the *physical interpretation* of the given Lagrangian. Although this question of interpretation may seem artificial in the case of (3), it is a serious matter for a Lagrangian which is poorly understood physically. As the above example illustrates, the underlying problem is that whenever Dirac's test fails, one has no preferred intrinsic way of determining—from the Lagrangian alone—the 'gauge equivalence class' of a given physical state. To assign a precise physical meaning to the Lagrangian under consideration, it is thus first necessary to specify—*by fiat*—these gauge equivalence classes. But such a specification is entirely equivalent to an *a priori* determination of the status of Dirac's conjecture: whether or not it fails and, if it fails, the extent to which it does so.

Thus, there is in general a choice of physical interpretation to be made. In order to obtain a consistent general theory of constrained dynamical systems, *it is imperative that this choice be part of the formal canonical analysis itself* rather than having to be made on a Lagrangian-by-Lagrangian basis. For example, one has the 'standard' interpretation in which all FCS constraints are *assumed* to be gauge, and the failure of Dirac's test is regarded as an indication of the existence of built-in gauge conditions.† The standard interpretation allows one to append the FCS constraints to the Hamiltonian as in (2) without changing the physical content of the theory.‡ As opposed to the standard viewpoint, one has the possibility of 'unorthodox' interpretations in which certain FCS constraints are *not* gauge (in particular, according to Cawley (1979) those which are not in G). Such constraints certainly cannot be included in the Hamiltonian. For ordinary electromagnetism, all interpretations coincide.

Which interpretation should one choose? To decide this, it is convenient to pass to the 'reduced' Hamiltonian formalism obtained by eliminating the physically irrelevant gauge variables from the theory. Dynamics is then played out on the reduced phase space (RPS) parametrised by the remaining ('true') dynamical variables and their conjugate momenta. By definition, the reduced formalism must satisfy the following criteria.

(i) There is a one-to-one onto correspondence between physically distinct states of the system and points of the RPS.

(ii) There exist *canonical* equations of motion on the RPS which, for given initial conditions, uniquely determine the time-development of the true dynamical degrees of freedom.

It can be shown (Gotay and Nester 1979a, Gotay 1979) that the standard interpretation is consistent with (i) and (ii), independent of the particular system under consideration. However, if one takes an unorthodox interpretation, this need *not* be the case, as the next example shows.

The canonical analysis of the Lagrangian

$$L = (1/2x)v_y^2 \tag{4}$$

(due to J Nester) gives two first-class constraints $p_x \approx 0$ and $-\frac{1}{2}(p_y)^2 \approx 0$ and the total Hamiltonian

$$H_T = (x/2)p_y^2 + up_x,$$

where u is arbitrary. The canonical equations are

$$\dot{x} = u, \qquad \dot{p}_x = 0, \qquad \dot{y} = 0, \qquad \dot{p}_y = 0.$$

Since $G = G_1 = \{p_x\}$ and $FC = \{p_x, p_y\}$, Dirac's test fails. In this case one has a choice of two possible interpretations.

According to the standard interpretation, the equivalent FCS constraint $p_y \approx 0$ generates a gauge transformation which induces an arbitrary evolution of the variable y (the equation $\dot{y} = 0$ being interpreted as a built-in gauge condition). The system is thus entirely gauge and the RPS consists of a single state which does not evolve in time. Conditions (i) and (ii) are satisfied, albeit trivially.

On the other hand, consider the unorthodox interpretation in which p_y does not generate a gauge transformation. The equations of motion then imply that only y is a true dynamical variable (p_y is not dynamic since it is constrained to vanish). The unorthodox RPS is therefore one-dimensional, i.e., the system has $\frac{1}{2}$ degrees of freedom! Consequently (ii) above is not satisfied for this system so that the reduced Hamiltonian formalism does not exist in any meaningful sense. Thus for Lagrangian (4), the unorthodox interpretation leads to physically absurd results.

† Of course, the standard interpretation *defines* exactly what is meant by a 'built-in gauge condition'.

‡ This result is also supported indirectly by the work of Dominici and Gomis (1980).

The above example is generic in the sense that typically, any unorthodox interpretation will: (a) lead to physical inconsistencies; and (b) violate property (i) and/or (ii) of the reduced Hamiltonian formalism thereby rendering this notion meaningless (Gotay and Nester 1979a, Gotay 1979). On the other hand, the standard interpretation is always consistent in the sense that it can never suffer from the above two defects. Since, as emphasised earlier, the choice of interpretation should be included in the general canonical analysis itself rather than being decided upon on a case-by-case basis, the above arguments show that the standard interpretation is the only viable one. But, according to this interpretation, Dirac's conjecture is *true*.

Finally, it is worth mentioning the role of the Euler–Lagrange equations in the canonical analysis. Cawley (1979) correctly points out that the Hamiltonian evolution of the system must be consistent with that predicted by the EL equations. However, it is also essential to take into account the fact that there may exist (the equivalent of) FCS constraints in the Lagrangian formalism (Gotay and Nester 1979b), thus necessitating an interpretation of Lagrangian dynamics as well. In particular, when comparing the Lagrangian and Hamiltonian dynamics of a system, it is important to take the same interpretation in both formalisms†.

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† Failure to do so can lead to spurious conclusions such as those reached by Cawley (1979), whose 'proof' of the failure of the Dirac conjecture arose as a result of comparing 'unorthodox' Lagrangian dynamics with 'standard' Hamiltonian dynamics.